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A CHARACTERIZATION OF THE CLASSICAL UNITALS

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A characterization of the classical unitals *)

by

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ABSTRACT

A characterization of the classical unitals is given in terms of certain geometrical properties.

KEY WORDS & PHRASES: *Unital, inversive plane, generalized quadrangle, Minkowski plane*

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1. INTRODUCTION

A *unital* or *unitary block design* is a $2 - (q^3+1, q+1, 1)$ design, i.e. an incidence structure of $q^3 + 1$ points, $q^2(q^2-q+1)$ lines, such that each line contains $q + 1$ points and any two distinct points are on a unique line. If q is a prime power, the absolute points and non-absolute lines of a unitary polarity of $PG(2,q^2)$ form a unital (see [2]). These unitals are called *classical*.

In [6], O'NAN showed that a classical unital satisfies the following condition.

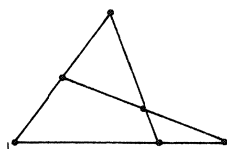(I) No four distinct lines intersect in six distinct points (see Figure 1).

No:

Fig. 1

In [5], PIPER conjectured that this property characterizes the classical unitals. Here we shall give a characterization for even q under the assumption that also the following condition holds.

(II) If L is a line, x a point not on L, M a line through x meeting L and and $y \neq x$ a point on M, then there exists a line $L' \neq M$ through y intersecting all lines through x which meet L.

To achieve this result we shall give another characterization for all q under the additional assumption that a third condition holds. To formulate this condition we need some notation. If x and y are distinct points, then we denote by xy the line through x and y. Given a point x, two lines L and L' missing x are called x-*parallel* (notation $L \parallel_x L'$) if and only if they intersect the same lines through x. Clearly, $\parallel_x$ is an equivalence relation on the set of lines missing x, and by (I) and (II), each equivalence class consists of q disjoint lines. Our third condition now reads as follows.

(III) Given a point x, three distinct lines M_1, M_2, M_3 through x and points y_i, z_i on M_i (i = 1,2,3) such that $(y_1y_2)\|_x(z_1z_2)$ and $(y_1y_3)\|_x(z_1z_3)$, then also $(y_2y_3)\|_x(z_2z_3)$.

Clearly, the presence of unitary transvections in PΓU(3,q) implies that the classical unitals satisfy conditions (II) and (III). In Section 2 we shall study unitals satisfying (I) and (II). Section 3 is devoted to the proof that unitals satisfying all three conditions are classical. Finally, in Section 4, we shall show that for even q, (III) is a consequence of (I) and (II).

2. UNITALS SATISFYING (I) AND (II)

Throughout this section U is a unital on $q^3 + 1$ points with point set X and line set L satisfying (I) and (II) above. If x ∈ X, then we denote by L^x the set of lines incident with x, and L_x will be the set of lines missing x. Furthermore, C_x will stand for the set of $\|_x$-equivalence classes on L_x. From [1] it is clear that we want to show that the incidence structure which has L^x as the set of points, C_x as the set of blocks and LIC (L ∈ L^x, C ∈ C_x) iff L meets one (hence all) lines of C, is the residual of an inversive plane of order q. We denote this incidence structure by $I^*(x) = (L^x, C_x)$. Clearly, $I^*(x)$ is a 2 - $(q^2, q+1, q)$ design.

LEMMA 1. *If x ∈ X and L,L' ∈ L_x such that L and L' both meet three distinct lines M_1, M_2, M_3 ∈ L^x, then L$\|_x$L', i.e. three distinct points of $I^*(x)$ are in at most one block of $I^*(x)$.*

PROOF. Let y ∈ M_1 ∩ L' and let L" be the line through y which is x-parallel to L, then L' ≠ L" contradicts (I). □

If M and M' are two distinct lines through a point x, then an easy counting argument shows that there are q -2 lines N_1,...,N_{q-2} through x such that no line of L_x meets M, M' and N_i, i = 1,...,q-2. Put $C^*(M,M') := \{M_1M'\} \cup \{N_1, N_2, ..., N_{q-2}\}$. We have to show that the $C^*(M,M')$ correspond to circles which will make $I^*(x)$ into an inversive plane. We have to show that N,N' ∈ $C^*(M,M') \Rightarrow C^*(M,M') = C^*(N,N')$. Clearly, $C^*(M,M') = C^*(M',M)$ and so it suffices to show that M" ∈ $C^*(M,M') \Rightarrow C^*(M,M') = C^*(M,M")$. This is the

contents of the next lemma.

LEMMA 2. *Fix a line* M $\in$ L *and two distinct points* x *and* y *in* M. *For*
M',M" $\in$ $L^{x}\backslash\{M\}$ *write* M'$\sim$M" *iff no line of* $L^{y}\backslash\{M\}$, *intersects both* M' *and* M"
or M' = M". *Then* $\sim$ *is an equivalence relation on* $L^{x}\backslash\{M\}$.

PROOF. For u,v $\in$ M let A^{*}(u,v) be the incidence structure with $L^{u}\backslash\{M\}$ as
points, $L^{v}\backslash\{M\}$ as lines, and incidence defined by PIB iff P and B meet
(P $\in$ $L^{u}\backslash\{M\}$, B $\in$ $L^{v}\backslash\{M\}$). If u, v, w are distinct points of M, then clearly
the mapping $\widetilde{\tau}^{u}_{v,w}$: A^{*}(u,v) $\to$ A^{*}(u,w) defined by

$$\widetilde{\tau}^{u}_{v,w}(P) := P, \qquad P \in L^{u}\backslash\{M\},$$

$$\widetilde{\tau}^{u}_{v,w}(B) := \text{u-parallel of B through w}, \qquad B \in L^{v}\backslash\{M\},$$

is an isomorphism of A^{*}(u,v) onto A^{*}(u,w). Now fix x,y $\in$ X, x $\neq$ y. If q > 2
and u, v are distinct points in M, u,v $\neq$ x,y, then

$$\widetilde{\delta}^{x,y}_{u,v} := \widetilde{\tau}^{y}_{v,x}\ \widetilde{\tau}^{v}_{u,y}\ \widetilde{\tau}^{u}_{x,v}\ \widetilde{\tau}^{x}_{y,u},$$

is an automorphism of A^{*}(x,y).

Now we claim that

(i)   For all u,v $\in$ M$\backslash\{x,y\}$, u $\neq$ v and for all P $\in$ $L^{x}\backslash\{M\}$, $\widetilde{\delta}^{x,y}_{u,v}(P) \neq$ P and
$\widetilde{\delta}^{x,y}_{u,v}(P) \sim$ P.

(ii) For all u,v,v' $\in$ M$\backslash\{x,y\}$, u $\neq$ v $\neq$ v' $\neq$ u and for all P $\in$ $L^{x}\backslash\{M\}$,
$\widetilde{\delta}^{x,y}_{u,v}(P) \neq \widetilde{\delta}^{x,y}_{u,v'}(P)$ and $\widetilde{\delta}^{x,y}_{u,v}(P) \sim \widetilde{\delta}^{x,y}_{u,v'}(P)$.

To prove these claims, write u P v for the u-parallel of P incident with
v. Then $\widetilde{\delta}^{x,y}_{u,v}(P)$ = y(u P v)x. Suppose y(u P v)x $\sim$ P or y(u P v)x = P. Then there
is a line L incident with x intersecting P and y(u P v)x. Then L intersects
u P v in a point a, say. Since au intersects P, we now have an O'Nan config-
uration on the lines M, P, L and au, contradicting (I).

Suppose y(u P v)x $\sim$ y(u P v')x or y(u P v)x = y(u P v')x. Let L be the
line through y intersecting y(u P v)x and y(u P v')x. Then L intersects u P v
and u P v' in points a and a', say. Since au intersects u P v', we have an
O'Nan configuration on M, L, u P v' and au, again in contradiction with (I).

For a given $P \in L^X \backslash \{M\}$, there are $q-2$ $Q \in L^X \backslash \{M\}$, $Q \neq P$ such that $Q \sim P$. Fixing u we can make $q-2$ choices for $v \in M \backslash \{x,y,u\}$. Thus, each $Q \in L^X \backslash \{M\}$, $Q \neq P$ can be written as $Q = \tilde{\delta}_{u,v}^{x,y}(P)$. If $Q = \tilde{\delta}_{u,v}^{x,y}(P) \sim P$ and $Q' = \tilde{\delta}_{u,v'}^{x,y}(P) \sim P$, then $Q \sim Q'$ by (ii). $\square$

Lemma 2 and its proof have a number of important corollaries.

COROLLARY 3. *Let* $x \in X$ *and let* ∞_x *be a new symbol. Put*

$$C^X := \{C^*(M,M') \cup \{\infty_x\} \mid M,M' \in L^X,\ M \neq M'\}.$$

Then $I(x) := (L^X \cup \{\infty_x\},\ C^X \cup C_x)$ *is an inversive plane of order* q *with point set* $L^X \cup \{\infty_x\}$ *and block set* $C^X \cup C_x$ *and incidence defined in the obvious way.*

PROOF. See the discussion preceding Lemma 2. $\square$

COROLLARY 4. *For* $x,y \in X$, $x \neq y$, *the incidence structure* $A^*(x,y)$ *of Lemma 2 is isomorphic to the derived design* $I(x)^{xy}$ *with* ∞_x *and the lines through* ∞_x *removed. The affine plane* $I(x)^{xy}$ *admits a dilatation group of order* $q-1$ *with centre* ∞_x.

PROOF. The automorphisms $\tilde{\delta}_{u,v}^{x,y}$ of $A^*(x,y)$ induce $q-2$ distinct nonidentity dilatations with centre ∞_x on $I(x)^{xy}$. Since $I(x)^{xy}$ has order q, these are the non-identity elements of the dilatation group with centre ∞_x of order $q-1$. $\square$

COROLLARY 5. *Let* $L \in L$ *and let* $x_1, x_2, \ldots, x_{q+1}$ *be the points on* L. *It is possible to partition the set of lines which meet* L *into classes* A_{ij}, $1 \leq i,j \leq q{+}1$, *such that for all* i *and* j

(i) $|A_{ij}| = q-1$

(ii) $M \in A_{ij} \Rightarrow x_i \in M$,

(iii) *every point* $x \in X \backslash L$ *is on exactly one line of* $\cup_k A_{kj}$,

(iv)   *no line which meets* L *in a point* $\neq x_i$, *meets two lines of* A_{ij},

(v)   *for all* k, i', $1 \leq k,i' \leq q{+}1$, $k \neq i,i'$ *and for all* $M \in A_{ij}$, *the* x_k-*parallel of* M *through* $x_{i'}$ *is in* $A_{i'j}$,

(vi)   *if* $1 \leq i' \leq q{+}1$, $i' \neq i$ *and* $M \in A_{ij}$, $M' \in A_{i',j}$ *then there exists a unique* $k \in \{1,\ldots,q{+}1\}$ *such that* M *and* M' *are* x_k-*parallel.*

PROOF. Consider $I(x_1)$. Number the circles of $I(x_1)$ through the two points ∞_{x_1} and L of $I(x_1)$ from 1 upto q+1. Apart from ∞_{x_1} and L each such circle contains (q-1) lines through x_1. These will be the sets $A_{1,j}$, $j = 1,\ldots,q+1$. For $i > 1$ and $1 \le j \le q+1$, let $A_{i,j}$ consist of the (q-1) lines through x_i which in $I(x_1)$ correspond to the (q-1) circles (not through ∞_{x_1}) in the pencil with carrier L and which contains circle j through ∞_{x_1} and L. Now (i) and (ii) are trivially satisfied. For (iii), note that the q+1 lines xx_i, $i = 1,\ldots,q+1$ are in A_{ij}'s with distinct j since the circles in a pencil with carrier L partition the set of points $\neq$ L of $I(x_1)$. To prove the other cases, observe that our subdivision of the set of lines meeting L into the classes A_{ij} would have remained the same if we had started by considering $I(x_i)$, $i > 1$ instead of $I(x_1)$. Thus, to prove (iv), it suffices to show that no line $M \in L^{x_1}\setminus\{L\}$ can intersect two distinct lines $N_1, N_2 \in A_{ij}$ with $i > 1$. This follows at once, since N_1 and N_2 correspond to tangent circles in $I(x_1)$. Also (v) is clear if we take k = 1 for then M and the x_k-parallel of M through x_i, represent the same circle in $I(x_1)$. Finally (vi) follows from (i), (v) and the easily shown fact two lines which meet L cannot be x_k- and x_ℓ-parallel for distinct k and ℓ.   □

Following PIPER [5], we are now able to associate with each line L of U an incidence structure $GQ(L)$ as follows. The points of $GQ(L)$ are the points $x \in X\setminus L$ and the sets A_{ij}, $1 \le i,j \le q+1$. The lines of $GQ(L)$ are the lines M of U meeting L, and 2(q+1) new lines, $A_1, A_2, \ldots, A_{q+1}$, $B_1, B_2, \ldots, B_{q+1}$. Incidence in $GQ(L)$ is defined as displayed in the following table.

	line of type M	line of type A_k or B_ℓ
point of type $x \in X\setminus L$	$x \in M$	never
point of type A_{ij}	$M \in A_{ij}$	i=k or j=ℓ

Incidence in $GQ(L)$

THEOREM 6. *Let* $U = (X,L)$ *be a unital with* q+1 *points on a line satisfying*
(I) *and* (II). *Then for each line* L $\in$ L, GQ(L) *is a generalized quadrangle*
with q+1 *points on a line and* q+1 *lines through a point. Moreover, any two*
nonintersecting lines m_1 *and* m_2 *of* GQ(L) *form a regular pair (in the sense*
of [7]) *provided* m_1 *and* m_2 *do not correspond to lines* M_1 *and* M_2 *of* U *such*
that $M_1 \in A_{ij}$ *and* $M_2 \in A_{k\ell}$ *with* i $\neq$ k *and* j $\neq$ ℓ. *In particular, the lines*
$A_1, \ldots, A_{q+1}, B_1, \ldots, B_{q+1}$ *of* GQ(L) *are regular.*

PROOF. Straightforward verification. $\square$

We shall see in Section 4 that if all lines of GQ(L) are regular, then
U is classical.

3. UNITALS SATISFYING (I), (II) AND (III)

Let $U = (X,L)$ be a unital satisfying (I), (II) and (III). Using (III)
it is easy to see that for any three distinct points x, y, z on a line L
there is a unique automorphism $\tau^x_{y,z}$ of U fixing x and all lines through x
and mapping y onto z: if u $\notin$ L then $\tau^x_{y,z}$(u) is defined to be the point of
intersection of xu and the x-parallel of yu through z, if v $\in$ L\{x}, fix a
point u $\notin$ L and define $\tau^x_{y,z}$(v) to be the point of intersection of L and
the x-parallel of uv through $\tau^x_{y,z}$(u).

THEOREM 7. *Let* $U = (X,L)$ *be a unital with* q+1 *points on a line satisfying*
(I), (II) *and* (III), *and let* G *be the automorphism group of* U *generated by*
the $\tau^x_{y,z}$. *Then* U *is classical,* G *is isomorphic to* $PSU(3,q^2)$ *and acts on* U
in the usual way.

PROOF. Clearly G is transitive on X. We claim that G acts 2-transitively on
X if q > 2 (the case q = 2 is left to the reader). To prove this, note that
the mappings $\tilde{\tau}^x_{y,z}$ of Lemma 2 are induced by the automorphisms $\tau^x_{y,z}$ of U.
Hence, also the mappings $\tilde{\delta}^{x,y}_{u,v}$ of Lemma 2 are induced by automorphisms
$\delta^{x,y}_{u,v} \in$ G of U. Since incidence in the inversive plane I(x) is determined by
incidence in U, $\delta^{x,y}_{u,v}$ induces an automorphism of I(x). By Corollary 4, this
is a dilatation of $I(x)^{xy}$ with centre ∞_x. Therefore, it can also be viewed
as a dilatation of $I(x)^{\infty_x}$ with centre xy. Thus in the affine plane $I(x)^{\infty_x}$,

each point is the centre of a dilatation. Hence $I(x)^{\infty x}$ is a translation plane and the group generated by the dilatations contains the full translation group of $I(x)^{\infty x}$ ([2, p.187]). Let $T(x)$ be the normal subgroup of G_x consisting of elements which induce (possibly identity) translations of $I(x)^{\infty x}$. Then $T(x)$ acts regularly on the points of $I(x)^{\infty x}$, i.e. on L^x, and for each line $L \in L^x$, $T(x)_L$ acts regularly on $L\backslash\{x\}$. Thus $T(x)$ is a normal subgroup of G_x acting regularly on $X\backslash\{x\}$, and G is 2-transitive. Applying [4] we get that G has a normal subgroup M such that $M \leq G \leq$ Aut M and M acts on X as one of the following groups in its usual 2-transitive representation: a sharply 2-transitive group, $PSL(2,q^3)$, $Sz(q^{3/2})$, $PSU(3,q^2)$, or a group of Ree type. Since $q^3 + 1 = (q+1)(q^2-q+1)$ is not a prime power for $q > 2$, the first alternative will not occur. If $H \leq G$ and x, y, z are three distinct points of X, then the H_{xy}-orbit of z is contained in xy, so has length $\leq q-1$. This excludes $M = PSL(2,q^3)$ and $M = Sz(q^{3/2})$. Moreover, this argument shows that if $M = PSU(3,q^2)$ then U is classical, for M_{xy} has a unique orbit of length $q-1$ on $X\backslash\{x,y\}$, all other orbits have length $(q^2-1)/(q+1,3)$ ([6, p. 499]). Now the $\tau^x_{y,z}$ can be identified with the unitary transvections and it follows that $G \simeq PSU(3,q^2)$. Thus we are left with the case that M is a group of Ree type. Since $q = 3^{2a+1}$, G contains an involution δ fixing at least two points $x,y \in X$ (Corollary 4). By [4], Lemma 3.3(v) and (ix), $\delta \in M$ and δ fixes $q+1$ points. Since δ is a dilatation on $I(x)^{\infty x}$ these must be the $q+1$ points of xy and so U is nothing but the Ree unital associated with M. Now, for $L \in L$, $<\delta> \times PSL(2,q) \simeq M_L \trianglelefteq G_L$ and so $<\tau^x_{y,z} \mid x,y,z \in L> \leq$ Aut$(PSL(2,q) = P\Gamma L(2,q)$, which shows that at least one, and hence all, $\tau^x_{y,z} \in M$, i.e. $G = M$ of order $(q^3+1)q^3(q-1)$. Now for a 3-Sylow group $T(x)$ of G, $T(x)/T(x)_L$ (x on L) is the elementary abelian translation group of $I(x)^{\infty x}$. Hence, for the derived group $T(x)^{(1)}$ of $T(x)$ we find $|T(x)^{(1)}| \leq |T(x)_L| = q$, contradicting Lemma 3.3(iii) of [4]. $\square$

4. MORE CHARACTERIZATIONS

Let $U = (X,L)$ be a unital satisfying (I) and (II). Consider the following two conditions.

(III') Given a point x and three distinct lines M_1, M_2, M_3 through x and points y_i, z_i on M_i ($i = 1,2,3$) such that $(y_1y_2)\parallel_x(z_1z_2)$, $(y_1y_3)\parallel_x(z_1z_3)$

and one of the lines $(y_i y_j)$ or $(z_i z_j)$ meets all three of M_1, M_2 and M_3, then $(y_2 y_3) \parallel_x (z_2 z_3)$.

(IV)   Given a point x and two distinct lines M_1 and M_2 through x and points y_1, y_3, z_1, z_3 on M_1, y_2, y_4, z_2, z_4 on M_2 such that $(y_1 y_2) \parallel_x (z_1 z_2)$, $(y_1 y_4) \parallel_x (z_1 z_4)$ and $(y_2 y_3) \parallel_x (z_2 z_3)$, then also $(y_3 y_4) \parallel_x (z_3 z_4)$.

Clearly, (III) implies (III') and (IV). The converse is also true.

LEMMA 8. *Let* $U = (X, L)$ *be a unital satisfying* (I), (II), (III') *and* (IV), *then also* (III) *holds.*

PROOF. Let x, M_i, y_i, z_i, i = 1,2,3 be as in (III). Suppose that M_1, M_2 and M_3 determine a circle in $I(x)$ not containing ∞_x, i.e. suppose there is a line through y_1 intersecting M_2 in u_2 and M_3 in u_3, say. Let v_2 (v_3) be the point of intersection of the x-parallel of $y_1 u_2$ through z_1 and M_2 (M_3). Using (III') we find that $(u_2 y_2) \parallel_x (v_2 z_2)$ and $(u_3 y_3) \parallel_x (v_3 z_3)$. Hence by (IV), $(y_2 y_3) \parallel_x (z_2 z_3)$ and (III) is shown to hold in this case. The remaining case is where M_1, M_2 and M_3 are on a circle of $I(x)$ containing ∞_x, i.e. no line of L_x meets all three of M_1, M_2 and M_3. Since the two circles of $I(x)$ corresponding to $y_1 y_2$ and $y_1 y_3$ cannot be tangent (for otherwise $y_1 y_2 = y_1 y_3$ and there is a line intersecting M_1, M_2 and M_3), there is a line M_4 through x which meets $y_1 y_2$ in y_4 and $z_1 z_2$ in z_4, say, and which also meets $y_1 y_3$ and $z_1 z_3$. Now looking at M_1, M_3 and M_4 are applying (III') we see that $(y_3 y_4) \parallel_x (z_3 z_4)$. Since M_2, M_3 and M_4 are not on a circle of $I(x)$ (for otherwise this would be the circle determined by M_1, M_2 and M_3), we can apply the previous case and conclude that $(y_2 y_3) \parallel_x (z_2 z_3)$.   □

The reason for considering (III') and (IV) is that in both cases there is a line M_i which is intersected by all lines mentioned in the condition. Thus, both (III') and (IV) have a (no doubt ackward) equivalent formulation $(\overline{\text{III'}})$ respectively $(\overline{\text{IV}})$ into terms of $GQ(M_i)$. Since the classical unital satisfies (III') and (IV), the classical generalized quadrangle $Q(4,q)$ on the points and lines of a hyperquadric in PG(4,q) must satisfy $(\overline{\text{III'}})$ and $(\overline{\text{IV}})$. So, conversely, if a unital U satisfying (I) and (II) has the property that $GQ(L)$ is isomorphic to $Q(4,q)$ for each line L of U, then U is classical.

THEOREM 9. *Let U be a unital with* q+1 *points on a line satisfying* (I) *and* (II). *If for each line* L *of* U, $G\!Q(L) \simeq Q(4,q)$, *i.e. if every line of* $G\!Q(L)$ *is regular, then* U *is classical.*

We are now in a position to prove that for even q, (I) and (II) suffice to characterize U.

THEOREM 10. *Let* $U = (X,L)$ *be a unital with* q+1 *points on a line satisfying* (I) *and* (II). *If* q *is even, then* U *is classical.*

PROOF. Let L be a line of U and let A_{ij}, $1 \le i,j \le q+1$ and A_i, B_i, $i = 1,\ldots$ $\ldots,q+1$ be defined as before. For each $x \in X \backslash L$ put

$$C(x) := \{A_{ij} \mid \exists \text{ line } M \in A_{ij} \text{ incident with } x\}.$$

By Corollary 5, $C(x)$ has exactly one point on each of the lines A_i and B_i, $i = 1,\ldots,q+1$. We claim that if $x,y \in X \backslash L$, $x \ne y$, then $|C(x) \cap C(y)| \le 2$. First suppose xy is a line meeting L, $xy \in A_{ij}$, say. then by Corollary 5(iv), $C(x) \cap C(y) = \{A_{ij}\}$. Now consider the case where xy is a line of U not meeting L. Suppose x_1, x_2, x_3 are distinct points of L such that $xx_1, yx_1 \in A_{1,1}$, $xx_2, yx_2 \in A_{2,2}$ and $xx_3, yx_3 \in A_{3,3}$. In $I(x)$, L, yx, yx_2, yx_3 correspond to circles with the following properties: yx_1, yx_2, yx_3 all go through the point xy of $I(x)$ and are tangent to L in respectively xx_1, xx_2 and xx_3. Since q is even, there is a point $\ne$ xy of $I(x)$ which is also on the circles yx_1, yx_2, yx_3, i.e. there is a line $M \ne xy$ through x intersecting yx_i, $i = 1,2,3$. By Lemma 1, $L \parallel_y M$ and so xy intersects L, a contradiction. We have shown that each triple A_{i_1,j_1}, A_{i_2,j_2}, A_{i_3,j_3} with $|\{i_1,i_2,i_3\}| = |\{j_1,j_2,j_3\}| = 3$ is covered at most once by a $C(x)$. Since there are $q^3 - q$ $C(x)$, each such triple is covered exactly once. Thus, with the A_{ij} as points, the A_i and B_i as lines and the $C(x)$ as circles, we have obtained a Minkowski plane $M(L)$ of even order q. By [3], $M(L)$ is isomorphic to the geometry of points, lines and plane sections of a quadric of index two in PG$(3,q)$. Since $G\!Q(L)$ is determined by $M(L)$ (the points of $G\!Q(L)$ correspond to the points and circles of $M(L)$, the lines of $G\!Q(L)$ correspond to the lines and pencils of $M(L)$, etc.) $G\!Q(L)$ is isomorphic to $Q(4,q)$ and so U is classical. $\quad\square$

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